Midterm 2

Question:	1	2	3	4	5	6	Total
Points:	15	10	10	10	10	10	65
Score:							

No books, no notes, no calculators.
Please show all your work, *except* for problem #1.
Simplify all answers. And please, read the problems carefully. Good luck!

Misc:

- $\mathbb{R}$ denotes the field of real numbers.
- $\mathbb{R}^n$ denotes the n-dimensional vector space.

1. This is the *only* problem where you don't need to show work.

   (a) (3 points) Suppose A is a 5-by-7 matrix, and has 4 pivot columns. What is the nullity (dimension of the null space) of A?

   ○ 0

   ○ 1

   ○ 2

   √ 3

   ○ 4

   ○ 5

   ○ 6

   ○ 7

   ○ It's impossible to tell unless one knows the specific form of A.

   (b) (12 points) Suppose V is a 4-dimensional vector space. Which of the following sets in V:

   - may or may not be a basis;
   - cannot be a basis;
   - must be a basis; or
   - are cannot exist?

   You don't know anything about the sets except the information in this question.

   Write the correct underlined words in the space provided.

   i. X, consisting of 3 vectors. **cannot**

   ii. Y, consisting of 4 vectors. **may or may not**

   iii. Z, consisting of 5 vectors. **may or may not**

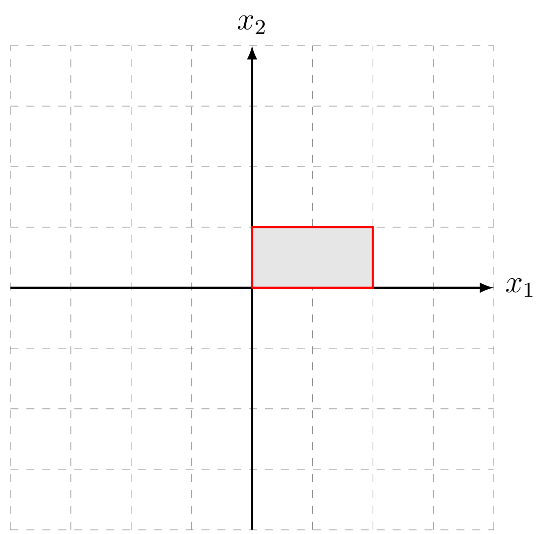
   iv. T, consisting of 3 linearly independent vectors. **cannot**

   v. U, consisting of 4 linearly independent vectors. **must**

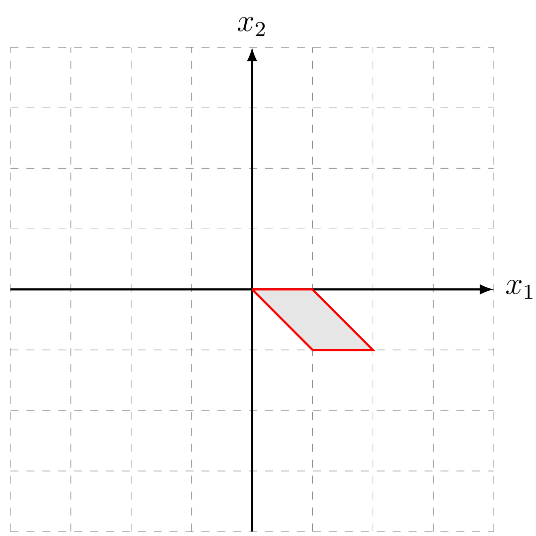
   vi. W, consisting of 5 linearly independent vectors. **cannot exist**

2. (10 points) Let $U = \{(x_1, x_2) \mid 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$ be the unit square in the (x_1, x_2)-plane (square with corners at $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$). What is the image of U under the linear transformation given by the matrix A?

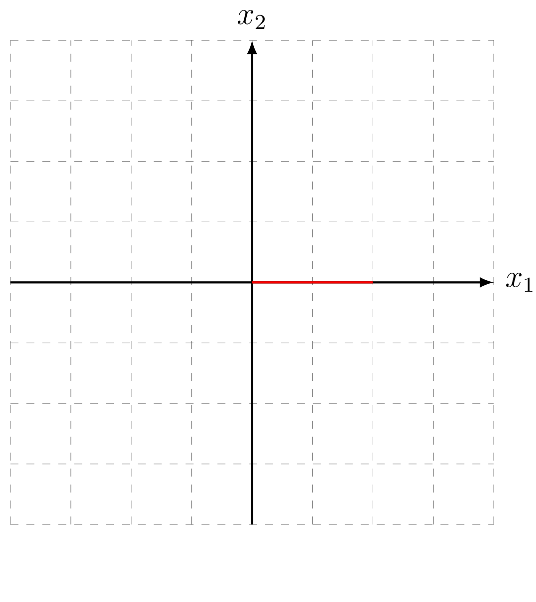
(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.



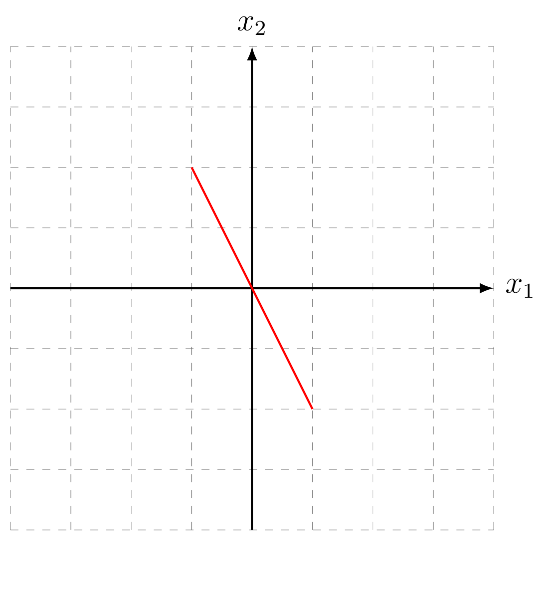
(b) $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.



(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.



(d) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$.



3. (10 points) Suppose $X = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$, where

$$\boldsymbol{c}_1 = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix},$$

and let $A = [\boldsymbol{c}_1 \ \boldsymbol{c}_2 \ \boldsymbol{c}_3]$:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

(a) Find an ordered basis for the column space of A.

Solution:

$$\begin{aligned} A &\sim \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R. \end{aligned}$$

Thus, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are the pivot columns, and a basis for the column space is

$$(\boldsymbol{c}_1, \boldsymbol{c}_2) = \left(\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \right).$$

(b) Find an ordered basis for the null space of A.

Solution:

$$\begin{aligned}
\mathsf{Null}\, A &= \{\boldsymbol{x} \mid A\boldsymbol{x} = \mathbf{0}\} \\
&= \{\boldsymbol{x} \mid R\boldsymbol{x} = \mathbf{0}\} \\
&= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_1 - x_3 = 0, \quad x_2 + 2x_3 = 0 \right\} \\
&= \left\{ \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} \middle| \; x_3 \in \mathbb{R} \right\} = \mathsf{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\},
\end{aligned}$$

so

$$\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$$

is a basis for the null space.

(c) Find an ordered basis for the $\mathsf{span}(X)$.

Solution: Since $\mathsf{span}(X)$ is the column space of A, from a, a basis for the column space is

$$(\boldsymbol{c}_1, \boldsymbol{c}_2) = \left(\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \right).$$

(d) Find $\boldsymbol{v} \in \mathbb{R}^3$, such that $\mathsf{span}(X \cup \{\boldsymbol{v}\}) = \mathbb{R}^3$.

Solution: Since the span does not change if we: (i) multiply a vector by a non-zero constant; (ii) add a multiple of one vector to another; and (iii) switch the order of vectors, we can perform elementary row operations on

$$\begin{bmatrix} 0 & 1 \\ 3 & 4 \\ 6 & 7 \end{bmatrix}^{\top} = \begin{bmatrix} 0 & 3 & 6 \\ 1 & 4 & 7 \end{bmatrix}$$

to get a simpler form of the span.

$$\begin{bmatrix} 0 & 3 & 6 \\ 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix},$$

thus

$$\mathsf{span}(X) = \mathsf{span}\left\{\boldsymbol{c}_1, \boldsymbol{c}_2\right\} = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Clearly, the vector

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

is not in the $\mathsf{span}(X)$, and $\mathsf{span}(X \cup \{\boldsymbol{v}\}) = \mathbb{R}^3$.

4. (10 points) Let $\boldsymbol{v}_1 = [-1, 2]^\top$, $\boldsymbol{v}_2 = [2, 1]^\top$. You can assume that $\boldsymbol{v}_1$ is perpendicular to $\boldsymbol{v}_2$, and $X = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ is a basis in $\mathbb{R}^2$. Let T be a reflection about $\boldsymbol{v}_1$.

(a) Let $\boldsymbol{z} = [1, 2]^\top$. Find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = K_X(\boldsymbol{z})$, the coordinate vector of $\boldsymbol{z}$ with respect to X.

Solution: Let $B = [\boldsymbol{v}_1\ \boldsymbol{v}_2] = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$. Then $K_X(\boldsymbol{z}) = B^{-1}\boldsymbol{z}$.

$$\begin{aligned}
[B \mid I] = \begin{bmatrix} -1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} -1 & 2 & 1 & 0 \\ 0 & 5 & 2 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} -5 & 0 & 1 & -2 \\ 0 & 5 & 2 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & -1/5 & 2/5 \\ 0 & 1 & 2/5 & 1/5 \end{bmatrix} = [I \mid B^{-1}].
\end{aligned}$$

Hence

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = K_X = \begin{bmatrix} -1/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}.$$

(b) Using the fact that $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$ and $T(\boldsymbol{v}_2) = -\boldsymbol{v}_2$, show that the coordinate vector of $T(\boldsymbol{z})$ is $\begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}$, i.e., $K_X(T(\boldsymbol{z})) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{c}$.

Solution: Since T is a linear map,

$$T(\boldsymbol{z}) = c_1 T(\boldsymbol{v}_1) + c_2 T(\boldsymbol{v}_1) = c_1 \boldsymbol{v}_1 - c_2 \boldsymbol{v}_1,$$

so

$$K_X(T(\boldsymbol{z})) = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{c}.$$

(c) Find a matrix A that corresponds to the transformation T, i.e., find a matrix A, such that $T(\boldsymbol{z}) = A\boldsymbol{z}$.

Solution: From b we have

$$K_X(T(\boldsymbol{z})) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{c}.$$

Since $K_X(T(\boldsymbol{z})) = B^{-1}T(\boldsymbol{z})$, and $\boldsymbol{c} = B^{-1}\boldsymbol{z}$,

$$B^{-1}T(\boldsymbol{z})) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B^{-1}\boldsymbol{z} \quad \Leftrightarrow \quad T(\boldsymbol{z}) = B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B^{-1}\boldsymbol{z},$$

in other words, $T(\boldsymbol{z}) = A\boldsymbol{z}$, where

$$A = B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B^{-1} = \left(\frac{1}{5}\right) \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}.$$

5. (10 points) Let $X = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, where $\boldsymbol{v}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2$, and $\boldsymbol{v}_2 = \boldsymbol{e}_2 + \boldsymbol{e}_3$.

(a) Show that X is linearly independent.

Solution:

$$A = [\boldsymbol{v}_1\ \boldsymbol{v}_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so $\mathsf{rank}(A) = 2 = |X|$, and X is linearly independent.

(b) Find a vector $\boldsymbol{v}_3$ which is not in the span of X.

Solution: Since the span does not change if we: (i) multiply a vector by a non-zero constant; (ii) add a multiple of one vector to another; and (iii) switch the order of vectors, we can perform elementary row operations on $A^\intercal$ to get a simpler form of the span:

$$A^\intercal = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

In other words, $\mathsf{span}(X) = \{[x, y, z]^\intercal \,|\, x, y \in \mathbb{R}, z = y - x\}$. Clearly, $\boldsymbol{v}_3 = [0, 0, 1]^\intercal = \boldsymbol{e}_3$ is not in the span of X.

(c) Let $Y = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Is Y a basis for $\mathbb{R}^3$?

Solution: Yes, it is, by independence extension lemma and since $|Y| = 3 = \mathsf{dim}\,\mathbb{R}^3$.

6. (10 points) An ordered basis for $\mathbb{R}^3$ is $B = (\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$, where

$$\boldsymbol{b}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \boldsymbol{b}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

(a) For the vector $\boldsymbol{a} = [4, 2, 3]^\top$, find the coordinate vector of $\boldsymbol{a}$ with respect to B.

Solution: Let $\boldsymbol{c} = [c_1, c_2, c_3]^\top$ be the coordinate vector. Then

$$\boldsymbol{a} = c_1\boldsymbol{b}_1 + c_2\boldsymbol{b}_2 + c_3\boldsymbol{b}_3 = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]\boldsymbol{c},$$

so $\boldsymbol{c} = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]^{-1}\boldsymbol{a}$:

$$\begin{aligned}[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3 \mid \boldsymbol{a}] = \begin{bmatrix}1 & 1 & 0 & 4\\1 & 0 & 1 & 2\\0 & 1 & 1 & 3\end{bmatrix} &\sim \begin{bmatrix}1 & 1 & 0 & 4\\0 & -1 & 1 & -2\\0 & 1 & 1 & 3\end{bmatrix}\\ &\sim \begin{bmatrix}1 & 0 & 1 & 2\\0 & -1 & 1 & -2\\0 & 0 & 2 & 1\end{bmatrix}\\ &\sim \begin{bmatrix}2 & 0 & 0 & 3\\0 & -2 & 0 & -5\\0 & 0 & 2 & 1\end{bmatrix}.\end{aligned}$$

We conclude that $K(\boldsymbol{a}) = \boldsymbol{c} = [3/2, 5/2, 1/2]^\top$.

(b) The coordinate vector with respect to B of $\boldsymbol{x}$, a vector in $\mathbb{R}^3$, is $K(\boldsymbol{x}) = [1, -1, 4]^\top$. What vector is $\boldsymbol{x}$?

Solution:

$$\begin{aligned}
\boldsymbol{x} &= (1)\,\boldsymbol{b}_1 + (-1)\,\boldsymbol{b}_2 + (4)\,\boldsymbol{b}_3 \\
&= [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]K(\boldsymbol{x}) \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}.
\end{aligned}$$

(c) The coordinate vector with respect to B of $\boldsymbol{y}$, a vector in $\mathbb{R}^3$, is $K(\boldsymbol{y}) = [-2, 2, 4]^\top$. Find the coordinate vector of $\boldsymbol{y}$ with respect to the standard basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.

Solution:

$$\boldsymbol{y} = (-2)\,\boldsymbol{b}_1 + (2)\,\boldsymbol{b}_2 + (4)\,\boldsymbol{b}_3 = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3]K(\boldsymbol{y}) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}.$$

Coordinates of any vector in the standard basis coincide with the components of that vector, since

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3.$$

We conclude that the coordinate vector of $\boldsymbol{y}$ with respect to the standard basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is $[0, 2, 6]^\top$.